Regge Poles, Power Series, and a Crossing-Symmetric Watson-Sommerfeld Transformation*†

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It is shown that there exists a close relationship between the analytic properties of the partial-wave amplitude as a function of complex-angular momentum l and those of the coefficients of expansions, other than the partial-wave expansion, as functions of the corresponding summation index, ν. The case of power series in z, and in the Mandelstam variables t and u is studied in detail. We show how the l-plane Regge poles for $\mathrm{Re}l>-\frac{1}{2}$ determine all the ν-plane poles for $\mathrm{Re}\nu>-\frac{1}{2}$ and vice versa. For the relativistic amplitude we write a representation consisting of three double power series in s, t, and u. We establish the analytic properties of the expansion coefficients in the two index variables which are implied by Regge analyticity in the l plane of each channel. This enables us to apply the Watson-Sommerfeld transformation twice and obtain a crossing-symmetric Regge-type representation which simultaneously displays the contributions of the Regge poles in all three channels.

I. INTRODUCTION

REGGE poles appear when one considers an analytic interpolation of the partial-wave scattering amplitude for complex values of angular momentum, l.[1] The position of these poles in the l plane depends on the energy and in potential scattering one has a clear and simple relation between the poles and bound states and resonances. In fact one of the main attractive features of Regge's work in potential scattering is the unified treatment it provides for bound states and resonances.

This feature and the simplicity of the asymptotic form of the scattering amplitude for large momentum transfers, has led several people to conjecture that results similar to Regge's hold for relativistic scattering amplitudes and in quantum field theory.[2] Some of these conjectures have been proved for certain simple models.[3] However, in the full theory the situation is far from clear and there are indications that the partial-wave amplitudes might have branch cuts as well as poles in the right-half l plane.

So far the discussion of Regge poles and their consequences has been closely tied to the Legendre expansion of the invariant amplitude. This has at least two drawbacks. The first is that the partial-wave expansion is not explicitly crossing-symmetric and consequently the resulting Regge representation obtained by applying the Watson-Sommerfeld transformation will also not display this symmetry. The situation becomes more serious when one wants to perform bootstrap-type calculations where one wants to impose both crossing symmetry and Regge behavior. In fact, it has not been known under what conditions are crossing symmetry and Regge behavior consistent with each other, if at all. All the models listed in Ref. 3 for which Regge properties have been proved are not crossing symmetric.

The second drawback of the partial-wave expansion in this context is that the partial-wave amplitudes are defined by a complicated transform of the absorptive part which involves Legendre functions of the second kind, Q_l.

The purpose of this paper is twofold. The first is an attempt to free the Regge analysis from strict association with partial-wave expansions. In fact we show that if Regge poles exist at all, essentially the same poles will appear when one starts with expansions other than the partial-wave expansion. We devote special attention to power series because of their simplicity. The possibility of using other expansions than the Legendre expansions enables us to achieve our second purpose which is to obtain a crossing-symmetric Regge formula. This is obtained by starting with double power series expansions in the Mandelstam variables s, t, and u.

In Sec. II we first prove two simple theorems. We show that if a function $f(z)=\sum a(l)P_l(z)$ is such that $a(l)$ is meromorphic in l for $\mathrm{Re}l>-\frac{1}{2}$, and as $|l|\to\infty$, $a(l)\sim\sqrt{l}e^{-l\xi}$, $\xi>0$, then if one expands $f(z)$ in a power series, $f(z)=\sum_\nu c(\nu)z^\nu$, $c(\nu)$ will again be meromorphic for $\mathrm{Re}\nu>-\frac{1}{2}$. It will have the same poles, α_j, as $a(l)$ plus poles at $\alpha_j-2, \alpha_j-4, \cdots$ etc. The converse of this theorem is also shown to be true. A similar result holds if one expands the nonrelativistic scattering amplitude in powers of momentum transfer t, $f(s,t)=\sum_\nu c'(\nu,s)t^\nu$.

In Sec. III we discuss briefly the relativistic problem and show that results similar to those in the preceding section hold. One starts in this case by writing the invariant amplitude as the sum of two power series, one in the Mandelstam variable t and the other in u. The coefficients of these series are given by simple Mellin transforms of the t and u absorptive parts, respectively. As the singularities of the partial-wave amplitudes in the l plane are closely related to the

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[1] T. Regge, Nuovo Cimento **14**, 951 (1959). See also A. Bottino, A. M. Longoni, and T. Regge, *ibid.* **23**, 954 (1962).
[2] For a complete list of references, see the review talk by S. D. Drell, in *Proceedings of the International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), pp. 897–911. In connection with the present paper see G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[3] B. W. Lee and R. Sawyer, Phys. Rev. **127**, 2266 (1962). S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).

singularities of these coefficients in the ν plane, one can either prove or disprove the conjectured Regge properties by studying these coefficients instead of the partial-wave amplitudes.

Before we go to the full crossing-symmetric problem, we discuss in Sec. IV the case of a function of two variables, s and t. We assume that the partial wave expansions of this function in either the s or t channel have properties similar to those proved by Regge for the partial-wave amplitude in potential scattering. Starting from a double power series expansion in s and t we show that the coefficient of such an expansion can be analytically continued in both indices and will satisfy the properties necessary to apply the Watson-Sommerfeld transformation twice. This will lead to a representation which exhibits the contributions of the Regge poles of both the s and t channels simultaneously.

The generalization of the results of Sec. IV to the case of the full relativistic amplitude is relatively simple. In Sec. V we start by writing the full amplitude as the sum of three double power series, one corresponding to each of the three main terms in the Mandelstam representation. The same steps as in Sec. IV follow for each of these series and we obtain an explicitly crossing symmetric Regge type representation. Finally, the conditions under which we obtained our result are briefly discussed.

II. REGGE POLES AND POWER SERIES EXPANSIONS

Regge poles show up when one considers the partial-wave expansion of a function $f(z)$,

$$f(z)=\sum_l a(l)P_l(z). \tag{1}$$

For a large class of scattering amplitudes in potential scattering it has been shown that the coefficients $a(l)$ have the following two properties.

(i) $a(l)$ has a unique interpolation which is meromorphic in the half-plane $\mathrm{Re}l>-\frac{1}{2}$, with a finite number of poles at $l=\alpha_j$, $\mathrm{Re}\alpha_j>-\frac{1}{2}$.

(ii) As $|l|\to\infty$ in the right- half plane $a(l)\sim c\sqrt{l}e^{-l\xi}$, where ξ is real and positive.

Suppose now one expands the function $f(z)$ in terms of other polynomials or to be more specific in power series in z,

$$f(z)=\sum_\nu c(\nu)z^\nu,\quad \nu=0,1,2,\cdots. \tag{2}$$

If $a(l)$ satisfies (i) and (ii) then it is clear that the series (2) will converge for $|z|<\cosh\xi$.

Two questions now naturally arise. First, is there a unique analytic extension of the coefficients $c(\nu)$ into the right-half ν plane, and second, what relation if any exists between the singularities of $c(\nu)$ and those of $a(l)$?

Clearly, if $f(z)\sim z^\alpha$ as $z\to\infty$ then, if $c(\nu)$ satisfies the conditions necessary for the application of the Sommerfeld-Watson transformation to (2), $c(\nu)$ will have a pole at $\nu=\alpha$ just as $a(l)$ has one at $l=\alpha$.[4] In this section we shall prove two theorems which show that there exists an even closer relationship between the singularities of $a(l)$ and those of $c(\nu)$.

Theorem 1. If $a(l)$ satisfies the conditions (i) and (ii) then there exists a unique interpolation of $c(\nu)$ which is meromorphic in the half-plane $\mathrm{Re}\nu>-\frac{1}{2}$. The poles of $c(\nu)$ will be the same set $\{\alpha_j\}$ as those of $a(l)$ with additional poles at

$$\alpha_j-2,\alpha_j-4,\cdots,\alpha_j-2n;\ \tfrac{3}{2}>\mathrm{Re}(\alpha_j-2n)>-\tfrac{1}{2}.$$

As $\mathrm{Re}\nu\to\infty$ in the half-plane, we have $c(\nu)\sim[\cosh\xi]^{-\nu}$. Also $c(\nu)$ vanishes as $|\mathrm{Im}\nu|\to\infty$, $\mathrm{Re}\nu>-\frac{1}{2}$.

Proof. From (i) and (ii) it follows that one can apply the Watson-Sommerfeld transformation to (1) and obtain in the usual manner

$$f(z)=\frac{i}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}dl\frac{a(l)}{\sin\pi l}P_l(-z)-\pi\sum_{j=1}^{N}\frac{\beta_jP_{\alpha_j}(-z)}{\sin\pi\alpha_j}. \tag{3}$$

If follows from (3) and (ii) that $f(z)$ is analytic in the cut z plane and we can write

$$f(z)=\frac{1}{\pi}\int_{z_0}^{\infty}\frac{D(z')}{z'-z}dz',\quad z_0>1, \tag{4}$$

where for the moment we have neglected to write the necessary subtractions. The lower limit z_0 is given by $z_0=\cosh\xi$ and ξ is defined in (ii).

We can now use (4) to get the following representation for the coefficients $c(\nu)$

$$c(\nu)=\frac{1}{\pi}\int_{z_0}^{\infty}D(z')z'^{-\nu-1}dz'. \tag{5}$$

This last expression allows us to define a unique interpolation of the coefficients $c(\nu)$ which is holomorphic in the region $\mathrm{Re}\nu>\mathrm{Re}\alpha_1$, where α_1 is the pole furthest to the right in the l plane, for it is clear from (3) that $D(z)\sim z^{\alpha_1}$ as $z\to\infty$. It is also evident from (5) that $c(\nu)\sim(z_0)^{-\mathrm{Re}\nu}$ as $\mathrm{Re}\nu\to\infty$ in the half-plane.

We can now continue $c(\nu)$ to the region $-\frac{1}{2}<\mathrm{Re}\nu\leq\mathrm{Re}\alpha_1$. To do that we note that using (3) we can write

$$D(z)=D_0(z)+\pi\sum_{j=1}^{N}\beta_jP_{\alpha_j}(z). \tag{6}$$

Here D_0 is the discontinuity of the background term and $D_0(z)\sim z^{-1/2}$ as $z\to\infty$. We, thus, obtain

$$c(\nu)=\frac{1}{\pi}\int_1^\infty D_0(z)z^{-\nu-1}dz+\sum_j\beta_j\int_1^\infty P_{\alpha_j}(z)z^{-\nu-1}dz. \tag{7}$$

[4] This remark holds for expansions in any set of polynomials $\phi_\nu(z)$ which are such that $\phi_\nu(z)\sim z^\nu$ as $z\to\infty$. The author is indebted to Professor M. Lévy for bringing to his attention the point that the leading pole must be the same in all such expansions.

For convenience we have set the lower limits of integration above to be unity instead of z_0. This does not change $c(\nu)$ since $D(z)$ is zero in the region $1\leq z\leq z_0$.

Now, the first term in (7) is regular in the region $\mathrm{Re}\nu>-\frac{1}{2}$. The integrals in the second term can be carried out and give[5]

$$\int_1^\infty P_\alpha(z)z^{-\nu-1}dz=2^{\nu-1}\Gamma\left(\frac{\nu+\alpha+1}{2}\right)\Gamma\left(\frac{\nu-\alpha}{2}\right)\Big/\pi^{\frac{1}{2}}\Gamma(\nu+1),\quad \mathrm{Re}\alpha>-\tfrac{1}{2}. \tag{8}$$

The right-hand side is obviously meromorphic in the region $\mathrm{Re}\nu>-\frac{1}{2}$ with poles $\nu=\alpha,\alpha-2,\cdots,\alpha-2n$; $-\frac{1}{2}<\mathrm{Re}(\alpha-2n)<\frac{3}{2}$. It is also easy to show from (7) and (8) that $c(\nu)\to 0$ as $|\mathrm{Im}\nu|\to\infty$. This completes the proof of our theorem.

The residues of the poles in the ν plane are related in a simple way to the β_j's and can be easily computed from (8).

It is clear now that $c(\nu)$ will satisfy the necessary conditions to enable us to apply the Watson-Sommerfeld transformation to the series (2). We obtain in the usual manner

$$f(z)=\frac{i}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\nu\frac{c(\nu)}{\sin\pi\nu}(-z)^\nu-\pi\sum_{j=1}^{N'}\frac{\gamma_j'(-z)^{\alpha_j'}}{\sin\pi\alpha_i'}, \tag{9}$$

where α_j' are the new poles and γ_j' are the new residues.

Each term in (9) has a cut starting from $z=0$ along the positive real axis. However, as we shall demonstrate later, one can easily show that the cuts of the first and second term cancel in the region $0\leq z<z_0$.

The question now arises about the converse of Theorem 1. Namely, if one starts with (2) and is given the fact that $c(\nu)$ has the following two properties:

(i′) $c(\nu)$ has a unique interpolarion which is meromorphic in the half-plane $\mathrm{Re}\nu>-\frac{1}{2}$, with a finite number of poles at $\nu=\alpha_j'$, $\mathrm{Re}\alpha_j'>-\frac{1}{2}$;

(ii′) as $\mathrm{Re}\nu\to\infty$, $c(\nu)\sim(z_0)^{-\nu}$ and as $|\mathrm{Im}\nu|\to\infty$, $c(\nu)\to 0$; what will be the properties of $a(l)$ in that case?

Theorem 2. If $c(\nu)$ satisfies the conditions (i′) and (ii′) then $a(l)$ will have a unique interpolation which is meromorphic in $\mathrm{Re}l>-\frac{1}{2}$ with poles at $\alpha_j',\alpha_j'-2,\cdots,\alpha_j'-2n$; $\mathrm{Re}(\alpha_j'-2n)>-\frac{1}{2}$. As $|l|\to\infty$, $a(l)\sim c\sqrt{l}e^{-l\xi}$.

Proof. The proof is very similar to that of Theorem 1. If (i′) and (ii′) hold then one can write (9) and this in turn will give us a dispersion relation in z as in (4). Using that we define $a(l)$ in the usual manner by

$$a(l)=\frac{(2l+1)}{\pi}\int_{z_0}^\infty Q_l(z)D(z)dz. \tag{10}$$

Clearly, this defines an analytic function in the region $\mathrm{Re}l>\mathrm{Re}\alpha_1'$, where α_1' is the pole with the greatest real part in the ν plane. To continue $a(l)$ to the region $\mathrm{Re}l<\mathrm{Re}\alpha_1'$ we use (9) to separate $D(z)$ into two terms as in (6). One term, coming from the background, gives a contribution regular in $\mathrm{Re}l>-\frac{1}{2}$, while the terms coming from the pole contributions in (9) will lead to integrals of the form,

$$\int_{z_0}^\infty Q_l(z)z^{\alpha'}dz=F(l,\alpha');\quad \mathrm{Re}\alpha'>-\tfrac{1}{2}. \tag{11}$$

This last integral can be performed[6] and the result is regular in the region $\mathrm{Re}l>-\frac{1}{2}$ except for poles at $l=\alpha'$, $\alpha'-2,\cdots,\alpha'-2n$. The easiest way to see this however is to use the expression for $Q_l(z)$ in terms of a hypergeometric function of argument $1/z^2$, and then make use of the hypergeometric series. The asumptotic behavior of $a(l)$ follows easily from (9) and (10) and is closely related to the behavior of $Q_l(z)$, $z\geq z_0$, for large $|l|$.

One detail which so far we have failed to mention in both Theorems 1 and 2 concerns the question of the uniqueness of the continuation to the left. Namely, in the case of Theorem 1 does $c(\nu)$ as defined in (7) and (8) coincide for integral values of $\nu=n$, with $n<\mathrm{Re}\alpha_1$, with the nth derivative of $f(z)$ divided by $n!$ and evaluated at $z=0$. Using (3) one can compute these derivatives and compare the result with (7) for $\nu=n$, $n<\mathrm{Re}\alpha_1$. The results are identical.

In the two above theorems we have for simplicity limited the discussion to cases where $a(l)$ and $c(\nu)$ have only poles in their corresponding right half-planes. However, it is not difficult to generalize Theorem 1 to the case where $a(l)$ has cuts of finite extension in the half-plane $\mathrm{Re}l>-\frac{1}{2}$. Corresponding cuts will also show up in the ν plane. The converse is also true.

At first sight Theorems 1 and 2 seem to be contradictory for the number of poles seems to increase if we go from the l plane to the ν plane and it also increases if we go from the ν plane to the l plane. However, one can easily see that there are cancellations which avoid this contradiction. For example if we start with $a(l)$ having one pole at $l=\alpha$ such that $\frac{3}{2}<\mathrm{Re}\alpha<\frac{7}{2}$, then in the ν plane we would have two poles $\alpha_1'=\alpha$ and $\alpha_2'=\alpha-2$. Now suppose we start in the ν plane with the poles $\nu=\alpha_1'$ and $\nu=\alpha_2'$ and go back to the l plane. Then we would have a pole at $l=\alpha_1'=\alpha$, but the residues will be such that the pole at $l=\alpha_1'-2=\alpha-2$ will just cancel with the pole at $l=\alpha_2'\equiv\alpha-2$. We thus get back to one pole at $l=\alpha$ in the l plane. Thus, it still makes sense to talk about the variable in which we have the least number of poles as the more natural one for the purposes of physical interpretation, if not for calculation.

Another question which naturally arises at this stage concerns the relationship, if any, of the singularities

[5] *Tables of Integral Transforms, Bateman Manuscript Project*, edited by A. Érdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 320, (3).

[6] See Ref. 5, p. 325, (26).

of $a(l)$ in the left-half plane, $\mathrm{Re}l<-\frac{1}{2}$, to those of $c(\nu)$ in the half-plane $\mathrm{Re}\nu<-\frac{1}{2}$. Do theorems similar to 1 and 2 hold for the left half-planes? The answer to this question is in the negative since one can give a fairly simple counter example. Consider the function $f(z)=(b-z)^{-1}$; $b>z$. We can write the two expansions

$$f(z)=\sum_l (2l+1)Q_l(b)P_l(z)\,,\qquad f(z)=\sum_\nu b^{-\nu-1}z^\nu. \tag{12}$$

Thus in this case $a(l)=(2l+1)Q_l(b)$ and $c(\nu)=b^{-\nu-1}$. In the right half-plane both have no poles. However, while $a(l)$ has an infinite number of poles at $l=-1,-2,\cdots,-n,\cdots$, in the region $\mathrm{Re}l<-\frac{1}{2}$, $c(\nu)$ is an entire function of ν and has no poles anywhere in the ν plane.

Theorems similar to 1 and 2 hold if instead of the power series expansion (2) we take expansions in terms of other polynomials, e.g., Gegenbauer polynomials, for example. However, in this paper we shall not go into the details of a general theorem giving the class of polynomials for which a result similar to Theorems 1 and 2 holds. For what follows we shall be mainly interested in an expansion similar to (2), a power series expansion in the momentum transfer variable.

In the above discussion we have suppressed the energy variable s. The amplitudes f are functions of s and the momentum transfer variable t, where $z=1+t/2s$. We can consider the expansion

$$f(s,t)=\sum_\nu c'(\nu,s)t^\nu. \tag{13}$$

If (i) and (ii) hold for $a(l,s)$ then (13) will converge for $|t|<t_0$ and $t_0=2s(\cosh\xi-1)$. A theorem similar to Theorem 1 will now hold for $c'(\nu,s)$ except in the present case the poles of c' will be at $\nu=\alpha_j,\alpha_j-1,\alpha_j-2,\cdots$, etc. The proof of such a theorem will be identical with that of Theorem 1. Following the same steps, we obtain an expression analogous to (7) for $c'(\nu,s)$

$$c'(\nu,s)=\frac{1}{\pi}\int_{t_0}^\infty D_0(1+t'/2s)t'^{-\nu-1}dt' + \sum_j \beta_i\int_{t_0}^\infty P_{\alpha_j}(1+t'/2s)t'^{-\nu-1}dt'. \tag{14}$$

The first term above is regular in the region $\mathrm{Re}\nu>-\frac{1}{2}$. One has only to study the integrals in the second term $I(\alpha,\nu)$,

$$I(\alpha,\nu)=\beta\int_{t_0}^\infty P_\alpha(1+t/2s)t^{-\nu-1}dt,\quad \mathrm{Re}\alpha>-\tfrac{1}{2}. \tag{15}$$

One can easily show that $I(\alpha,\nu)$ is regular in ν in the region $\mathrm{Re}\nu>-\frac{1}{2}$ except for poles at $\nu=\alpha,\alpha-1,\cdots,\alpha-n$; where the integer n is defined by $\frac{1}{2}>\mathrm{Re}(\alpha-n)>-\frac{1}{2}$. This result follows most directly from the fact that $P_\alpha(x)$, $x>1$, $\mathrm{Re}\alpha>-\frac{1}{2}$, can be expanded in such a way as to factor out all the terms that grow faster than $x^{-1/2}$ as $x\to\infty$. Namely, one has

$$P_\alpha(x)=g_0(\alpha)x^\alpha+g_1(\alpha)x^{\alpha-2}+\cdots+g_{n'}(\alpha)x^{\alpha-2n'}+G_\alpha(x). \tag{16}$$

Here the integer n' is determined by the condition

$$\tfrac{3}{2}>\mathrm{Re}(\alpha-2n')>-\tfrac{1}{2},$$

and the function $G_\alpha(x)$ decreases at least as fast as $x^{-1/2}$ as $x\to\infty$. One can derive (16) from the expression for $P_\alpha(x)$ in terms of hypergeometric functions of argument $1/x^2$.[7] We can now substitute (16) in (15) and obtain

$$I(\alpha,\nu)=I_0(\alpha,\nu)+\frac{\gamma_0 t_0^{-(\nu-\alpha)}}{\nu-\alpha}+\frac{\gamma_1 t_0^{-(\nu-\alpha+1)}}{\nu-\alpha+1}+\cdots+\frac{\gamma_n t_0^{-(\nu-\alpha+n)}}{\nu-\alpha+n};\qquad \tfrac{1}{2}>\mathrm{Re}(\alpha-n)>-\tfrac{1}{2}. \tag{17}$$

Here $I_0(\alpha,\nu)$ is regular in the region $\mathrm{Re}\nu>-\frac{1}{2}$. The new residues, γ_j, are all proportional to β. In the Appendix we give the expressions for the first few γ_j's in terms of β and α.

It is easy to verify from (14) and (17) that as $\mathrm{Re}\nu\to\infty$, $c'(\nu,s)\sim t_0^{-\nu}$ and that as $|\mathrm{Im}\nu|\to\infty$, $c'(\nu,s)$ vanishes if $\mathrm{Re}\nu>-\frac{1}{2}$.

We shall now write down the Watson-Sommerfeld transformation for (13) in order to show explicitly how one can write the pole contributions in such a way that they exhibit the right cuts in the t plane as was accomplished for the usual Regge poles in an earlier paper of the author.[8] For simplicity let us assume we have only one pole in the l plane with $\mathrm{Re}\alpha(s)>-\frac{1}{2}$. For $|t|<t_0$ we can carry out the Watson-Sommerfeld transformation on (13) and obtain

$$f(s,t)=\frac{i}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{c'(\nu,s)}{\sin\pi\nu}(-t)^\nu d\nu-\pi\sum_{j=0}^{n}\frac{\gamma_j(s)(-t)^{\alpha-j}}{\sin\pi(\alpha-j)},\qquad \tfrac{1}{2}>\mathrm{Re}(\alpha-n)>-\tfrac{1}{2}. \tag{18}$$

When $\mathrm{Re}\alpha\approx n'$, where n' is any integer, and $\mathrm{Im}\alpha$ is small, then by substituting the expressions for the γ_j's given in the Appendix the sum above reduces to a Regge pole term of the usual form, i.e., $-\pi[\beta P_\alpha(-z)/\sin\pi\alpha]$.

Both terms in (18) define a function of t which is analytic in the t plane except for a cut along the positive real axis. As was done in Ref. 8 we shall show that the cuts of the two terms in (18) actually cancel in the region $0\leq t<t_0$. To do that we note that in the strip

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[7] *Higher Transcendental Functions*, Bateman Manuscript Project, edited by A. Érdelyi (McGraw Hill Book Company, Inc., New York, 1953), Vol. 1, p. 126, (23).
[8] N. N. Khuri, Phys. Rev. **130**, 429 (1963).

$-1<\text{Re}\nu<0$ the function $(t)^\nu$ has the following representation,

$$\frac{t^\nu}{\sin\pi\nu}=-\frac{1}{\pi}\int_0^\infty \frac{x^\nu}{x+t}dx\,,\quad -1<\text{Re}\nu<0. \tag{19}$$

We can substitute this representation in the integrand of the first term in (18) and obtain

$$f(s,t)=\int_0^\infty dx\frac{b(x,s)}{x-t-i\epsilon}-\pi\sum_{j=0}^{n}\gamma_j(s)\frac{(-t)^{\alpha-j}}{\sin\pi(\alpha-j)}, \tag{20}$$

where

$$b(x,s)=\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} x^\nu c'(\nu,s)d\nu\,. \tag{21}$$

Now we know that $c'(\nu,s)\sim(t_0)^{-\nu}$ for large Reν. Thus, we can perform the integration in (21) for $x<t_0$ and obtain by closing the contour to the right

$$b(x,s)=-\sum_{j=0}^{n}\gamma_j(s)x^{\alpha-j},\quad x<t_0. \tag{22}$$

This enables us to rewrite (20) as

$$f(s,t)=\int_{t_0}^\infty dx\frac{b(x,s)}{x-t-i\epsilon} -\sum_{j=0}^{n}\gamma_j(s)\left[\int_0^{t_0}\frac{x^{\alpha-j}}{x-t-i\epsilon}dx+\frac{\pi(-t)^{\alpha-j}}{\sin\pi(\alpha-j)}\right], \tag{23}$$

where we must remember that $\text{Re}(\alpha-j)>-\frac{1}{2}$. The first term has now the proper cut. Similarly one can easily check that for the two terms in the brackets the discontinuities in the region $0\leq t<t_0$ cancel each other. We must stress here that in the discussion (18)–(23) we have chosen that branch of the function $(-t)^\nu$ for which the cut is on the positive real axis.

From (23) we can now identify the full contribution of a pole in the ν plane at $\nu=\alpha-j$ and express it as

$$R(t;\alpha(s)-j)=-\gamma_j(s)\left[\int_0^{t_0}\frac{x^{\alpha-j}}{x-t-i\epsilon}dx+\frac{\pi(-t)^{\alpha-j}}{\sin\pi(\alpha-j)}\right],\quad \text{Re}(\alpha-j)>-\tfrac{1}{2}. \tag{24a}$$

The contribution of a pole in the left half-plane, $\text{Re}(\alpha-j)<-\frac{1}{2}$ would be given by

$$R(t;\alpha(s)-j)=\gamma_j(s)\int_{t_0}^\infty\frac{x^{\alpha-j}}{x-t-i\epsilon}dx\,,\quad \text{Re}(\alpha-j)<-\tfrac{1}{2}. \tag{24b}$$

In the strip, $-1<\text{Re}(\alpha-j)<0$, the representations (24a) and (24b) are identical. Finally, we write one more result which will be useful in Sec. IV,

$$R(t;\alpha-j)=\sum_\nu c(\nu;\alpha-j)t^\nu,$$

where

$$c(\nu;\alpha-j)=\gamma_j(s)\frac{t_0^{-(\nu-\alpha+j)}}{\nu-\alpha+j}. \tag{25}$$

If $\text{Re}(\alpha-j)>-\frac{1}{2}$ the sum of the series (25) gives (24a) and if $\text{Re}(\alpha-j)<-\frac{1}{2}$ it gives (24b).

III. POWER SERIES EXPANSIONS IN THE RELATIVISTIC PROBLEM

Before we discuss the crossing-symmetric Watson-Sommerfeld transformation, we shall in this section briefly define, for the relativistic problem, amplitudes corresponding to $c'(\nu,s)$ of the previous section and point out the relation between their singularities and those of the usual partial wave amplitudes.

We consider elastic scattering of two spin zero particles with equal mass, $m=1$. The invariant amplitude, $A(s,t)$, we assume satisfies a dispersion relation for fixed s,

$$A(s,t)=\frac{1}{\pi}\int_4^\infty\frac{A_t(s,t')}{t'-t}dt'+\frac{1}{\pi}\int_4^\infty\frac{A_u(s,u')}{u'-u}du'. \tag{26}$$

We can expand each of the two terms above in power series and write

$$A(s,t)=\sum_\nu c_1(\nu,s)t^\nu+\sum_\nu c_2(\nu,s)u^\nu, \tag{27}$$

where

$$c_1(\nu,s)=\frac{1}{\pi}\int_4^\infty A_t(s,t)t^{-\nu-1}dt,\qquad c_2(\nu,s)=\frac{1}{\pi}\int_4^\infty A_u(s,u)u^{-\nu-1}du. \tag{28}$$

The series (27) converge for s, t, and u lying inside the small Mandelstam triangle, i.e., the Euclidean region. The usual partial-wave amplitudes are defined by formulas similar to (28), namely,[9]

$$a_{(1)}(l,s)=\frac{1}{\pi}\int_4^\infty\frac{dt}{2q^2}Q_l\left(1+\frac{t}{2q^2}\right)A_t(s,t'),\qquad a_{(2)}(l,s)=\frac{1}{\pi}\int_4^\infty\frac{du}{2q^2}Q_l\left(1+\frac{u}{2q^2}\right)A_u(s,u). \tag{29}$$

Here $4q^2=s-4$ and the amplitudes of even- or odd-l parity, $a_\pm(l,s)$, are related to $a_{(1)}$ and $a_{(2)}$ by

$$a_\pm(l,s)=a_{(1)}(l,s)\pm a_{(2)}(l,s). \tag{30}$$

[9] See, for example, V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1962 (1961) [translation: Soviet Phys.—JETP **14**, 1395 (1962)].

One can now easily prove theorems similar to those in the previous section relating the singularities of $c_1(\nu,s)$ to those of $a_{(1)}(l,s)$ and vice-versa. Of course similar results hold for $c_2(\nu,s)$ and $a_{(2)}(l,s)$. We shall not do this here since it involves a trivial repetition of the previous section. We would like to make only two remarks.

First, the expressions in (28) are much simpler than those in (29) and one might as well study the analytic properties of c_1 and c_2 in the ν plane. If for example they turn out to have only poles for $\mathrm{Re}\nu>-\frac{1}{2}$ then $a_{(1)}$ and $a_{(2)}$ will have only poles in the right-half l plane.[10] Second, the argument of Gribov and Pomeranchuk leading to the conclusion that $a_{(1),(2)}$ must have an essential singularity at $l=-1$ does not automatically apply to c_1 and c_2. An optimist could hope that the trouble discovered by Gribov and Pomeranchuk is kinematic in nature and related to the poles of the Legendre function Q_l for negative integral values of l, and that $c_1(\nu,s)$ and $c_2(\nu,s)$ do not have the same difficulty.

IV. DOUBLE WATSON-SOMMERFELD TRANSFORMATION

In this section we shall consider a special nonphysical case of a function of two variables, $f(s,t)$, and show how under certain assumptions one can write a double Watson-Sommerfeld transformation which will exhibit the Regge poles of both channels simultaneously. We do this at this stage because it will make the discussion of the full relativistic crossing-symmetric case in the next section easier and clearer.

Let us start with a function $f(s,t)$ which satisfies the following representation,

$$f(s,t)=\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{\rho(s',t')}{(s'-s)(t'-t)}. \qquad (31)$$

Here we have again neglected to write down subtractions, but as we shall see below these will not be necessary for our discussion. We stress again that in all this section $f(s,t)$ is just a mathematical object having certain properties and is not an actual scattering amplitude.

We now define the two variables z_1 and z_2,

$$z_1=1+2t/(s-4), \quad z_2=1+2s/(t-4). \qquad (32)$$

In terms of these variables one can write down the following two partial-wave expansions

$$f(s,t)=\textstyle\sum_l(2l+1)a_1(l,s)P_l(z_1), \qquad (33)$$

or

$$f(s,t)=\textstyle\sum_l(2l+1)a_2(l,t)P_l(z_2). \qquad (34)$$

From (31) it follows that (33) will converge in an ellipse in the z_1 plane centered at the origin and with semimajor axis $z_1=1+8/(s-4)$. A corresponding ellipse exists for (34).

Let us further restrict the function $f(s,t)$ by assuming that both $a_1(l,s)$ and $a_2(l,t)$ are meromorphic in l in the region $\mathrm{Re}l>-\frac{1}{2}$. For simplicity we take the case where both a_1 and a_2 have only one pole each which for some real values of s or t shows up in the region $\mathrm{Re}l>-\frac{1}{2}$. For $a_1(l,s)$ we have a pole at $l=\alpha_1(s)$ such that

$$\mathrm{Re}\alpha_1(s)>-\tfrac{1}{2}, \quad s_0<s<s_1. \qquad (35)$$

Similarly for $a_2(l,t)$ we take a pole at $l=\alpha_2(t)$ such that

$$\mathrm{Re}\alpha_2(t)>-\tfrac{1}{2}, \quad t_0<t<t_1. \qquad (36)$$

For values of s and t outside the intervals given in (35) and (36) both α_1 and α_2 move into the left-half plane. We take s_1, $t_1>4$ and s_0, $t_0<4$. This last inequality can be relaxed and one can take s_0, $t_0>4$ but it complicates the algebra below. The first inequality, s_1, $t_1>4$, is essential. What we are requiring is that any pole which shows up for some value of s (or t) in the region $\mathrm{Re}l>-\frac{1}{2}$ should eventually, as we increase $s(t)$, move back into the region $\mathrm{Re}l<-\frac{1}{2}$. If $\alpha_1(s)$ is real and has a positive slope below threshold, $s<4$, then it is clear that $s_1>4$. We again stress that we are taking only one pole in each channel just to simplify the algebra. The argument below could be carried through for any finite number of poles if each pole satisfies conditions similar to (35) or (36).

Now, finally, we shall assume that for large $|l|$ both $a_1(l,s)$ and $a_2(l,t)$ have the following asymptotic behavior similar to (ii):

$$a_1(l,s)\sim e^{-(l+\frac{1}{2})\xi(s)}/\sqrt{l}; \quad a_2(l,t)\sim e^{-(l+\frac{1}{2})\xi(t)}/\sqrt{l}, \qquad (37)$$

where

$$\cosh\xi(s)=1+8/(s-4).$$

In brief we have assumed that both $a_1(l,s)$ and $a_2(l,t)$ have properties very similar to those found by Regge for the partial-wave amplitude in potential scattering. We are interested in the implications of such properties for other expansions.

We go back at this point to the function $f(s,t)$. We can write a double-power series expansion for this function

$$f(s,t)=\sum_{\nu,\mu}c(\nu,\mu)s^\nu t^\mu; \quad \nu,\mu=0,1,2,\cdots. \qquad (38)$$

This series will converge absolutely for $|s|$, $|t|<4$. Again we ask ourselves the natural question: can one find a unique analytic interpolation of $c(\nu,\mu)$ regular in both ν and μ and having the properties necessary for performing a double Watson-Sommerfeld transformation? The answer to this question is in the affirmative and we shall show in the rest of this section that one can obtain a double Regge representation which will simultaneously exhibit the poles of both the s and the t channels.

[10] Such a suggestion has already been followed by N. Nakanishi. He considers a certain subclass of diagrams in a ϕ^3 theory and by studying c_1 and c_2 shows that for that subclass the partial-wave amplitude is holomorphic for $\mathrm{Re}l>0$. See N. Nakanishi (unpublished).

From (31) we obtain the following expression for $c(\nu,\mu)$,

$$c(\nu,\mu)=\frac{1}{\pi^2}\int_4^\infty ds\int_4^\infty dt\,\rho(s,t)s^{-\nu-1}t^{-\mu-1}. \qquad (39)$$

Regardless of the subtractions in (31) the integrals above converge if $\mathrm{Re}\nu$ and $\mathrm{Re}\mu$ are large enough. In fact, it is easy to see that the right-hand side defines a unique analytic interpolation of $c(\nu,\mu)$ holomorphic in the region

$$\begin{aligned}\mathrm{Re}\nu&>\sup_t \mathrm{Re}\alpha_2(t),\\ \mathrm{Re}\mu&>\sup_s \mathrm{Re}\alpha_1(s).\end{aligned} \qquad (40)$$

We shall now show that the properties we have assumed for $a_1(l,s)$ and $a_2(l,t)$ will enable us to enlarge the domain (40) and to factor out the singular terms.

To do this we have first to derive some properties of $\rho(s,t)$. Under our assumptions about a_1 and a_2 we can perform the Watson-Sommerfeld transformation on (33) and (34). We obtain two representations for $f(s,t)$ similar to (3). If we now take the double discontinuity of each of these representations we obtain the following two expressions for $\rho(s,t)$:

$$\rho(s,t)=\sigma_1(s,t)+\frac{\pi}{2i}\left[\beta_1(s)P_{\alpha_1}\left(1+\frac{2t}{s-4}\right)-\beta_1^*(s)P_{\alpha_1^*}\left(1+\frac{2t}{s-4}\right)\right],\quad 4<s<s_1,\ t>4. \qquad (41)$$

$$\rho(s,t)=\sigma_2(s,t)+\frac{\pi}{2i}\left[\beta_2(t)P_{\alpha_1}\left(1+\frac{2s}{t-4}\right)-\beta_2^*(t)P_{\alpha_2^*}\left(1+\frac{2s}{t-4}\right)\right],\quad 4<t<t_1;\ s>4. \qquad (42)$$

Here σ_1 and σ_2 come from the discontinuity of the background term and

$$\begin{aligned}\sigma_1(s,t)&\sim t^{-1/2} \quad \text{as} \quad t\to\infty,\\ \sigma_2(s,t)&\sim s^{-1/2} \quad \text{as} \quad s\to\infty.\end{aligned} \qquad (43)$$

For $s>s_1$ (or $t>t_1$) no pole term appears and one has

$$\begin{aligned}\rho(s,t)&\sim t^{-1/2}, \quad s>s_1, \quad t\to\infty;\\ \rho(s,t)&\sim s^{-1/2}, \quad t>t_1, \quad s\to\infty.\end{aligned} \qquad (44)$$

To factor out the singular terms in (39) we rewrite it as follows

$$\begin{aligned}c(\nu,\mu)=&\frac{1}{\pi^2}\int_4^{s_1} ds\int_4^{t_1} dt\,\rho(s,t)s^{-\nu-1}t^{-\mu-1}\\ &+\frac{1}{\pi^2}\int_4^{s_1} ds\int_{t_1}^\infty dt\,\rho(s,t)s^{-\nu-1}t^{-\mu-1}\\ &+\frac{1}{\pi^2}\int_4^{t_1} dt\int_{s_1}^\infty ds\,\rho(s,t)s^{-\nu-1}t^{-\mu-1}\\ &+\frac{1}{\pi^2}\int_{s_1}^\infty ds\int_{t_1}^\infty dt\,\rho(s,t)s^{-\nu-1}t^{-\mu-1}.\end{aligned} \qquad (45)$$

The first term above defines a function which is holomorphic in the region $\mathrm{Re}\nu>-\frac{1}{2}$, $\mathrm{Re}\mu>-\frac{1}{2}$. Similarly, it follows from (44) that the fourth term in (45) is also holomorphic in the same region. Thus, the only singularities in the right-half planes come from the second and third terms in (45). In the second term we can use the representation (41) for ρ and in the third we use (42). From (43) it follows that the terms with σ_1 and σ_2 do not give any singularities in the region, $\mathrm{Re}\nu>-\frac{1}{2}$, $\mathrm{Re}\mu>-\frac{1}{2}$, and, thus, (45) can be rewritten as

$$\begin{aligned}c(\nu,\mu)=c_0'(\nu,\mu)&+\frac{1}{\pi}\int_4^{s_1} ds\int_{t_1}^\infty dt\,s^{-\nu-1}t^{-\mu-1}\left(\frac{1}{2i}\right)\\ &\times\left[\beta_1(s)P_{\alpha_1}\left(1+\frac{2t}{s-4}\right)-\beta_1^*(s)P_{\alpha_1^*}\left(1+\frac{2t}{s-4}\right)\right]\\ &+\frac{1}{\pi}\int_4^{t_1} dt\int_{s_1}^\infty ds\,s^{-\nu-1}t^{-\mu-1}\left(\frac{1}{2i}\right)\\ &\times\left[\beta_2(t)P_{\alpha_2}\left(1+\frac{2s}{t-4}\right)-\beta_2^*(t)P_{\alpha_2^*}\left(1+\frac{2s}{t-4}\right)\right].\end{aligned} \qquad (46)$$

The function $c_0'(\nu,\mu)$ is now holomorphic in the region $\mathrm{Re}\nu>-\frac{1}{2}$, $\mathrm{Re}\mu>-\frac{1}{2}$. Using the methods of Sec. II we can now easily investigate the singularities coming from the two integrals in (46). However, before we do that we reduce (46) to a more convenient form. We assume that both β_1 and β_2 vanish for large values of their respective arguments. This assumption will be needed at other points in the discussion and we might as well also use it now. By adding and subtracting terms which are also regular in $\mathrm{Re}\nu>-\frac{1}{2}$, $\mathrm{Re}\mu>-\frac{1}{2}$, to the right-hand side of (46) we can change it to the following form

$$\begin{aligned}c(\nu,\mu)=c_0''(\nu,\mu)&+\frac{1}{\pi}\int_4^{s_1} ds\int_4^\infty dt\,s^{-\nu-1}t^{-\mu-1}\left(\frac{1}{2i}\right)\\ &\times\left[\beta_1(s)P_{\alpha_1}\left(1+\frac{2t}{s-4}\right)-\beta_1^*(s)P_{\alpha_1^*}\left(1+\frac{2t}{s-4}\right)\right]\\ &+\frac{1}{\pi}\int_4^{t_1} dt\int_4^\infty ds\,s^{-\nu-1}t^{-\mu-1}\left(\frac{1}{2i}\right)\\ &\times\left[\beta_2(t)P_{\alpha_2}\left(1+\frac{2s}{t-4}\right)-\beta_2^*(t)P_{\alpha_2^*}\left(1+\frac{2s}{t-4}\right)\right].\end{aligned} \qquad (47)$$

Here again $c_0''(\nu,\mu)$ is holomorphic in the domain $\mathrm{Re}\nu>-\frac{1}{2}$, $\mathrm{Re}\mu>-\frac{1}{2}$. The integrals in (47) have the same singularities in this same domain as those in (46).

One of the integrations in each term can be performed as was done in (15) and (17). Using (17) we get

$$c(\nu,\mu)=c_0(\nu,\mu)+c(\nu,\mu;\alpha_1)+c(\nu,\mu;\alpha_2), \qquad (48)$$

where[11]

$$c(\nu,\mu;\alpha_1)=\sum_{j=0}^{n}\frac{1}{2\pi i}\int_4^\infty ds'\,s'^{-\nu-1}\left[\frac{(4)^{-(\mu-\alpha_1+j)}\gamma_{1j}(s')}{\mu-\alpha_1(s')+j}-\frac{(4)^{-(\mu-\alpha_1^*+j)}\gamma_{1j}^*(s')}{\mu-\alpha_1^*(s')+j}\right], \quad (49)$$

$$c(\nu,\mu;\alpha_2)=\sum_{j=0}^{n'}\frac{1}{2\pi i}\int_4^\infty dt'\,t'^{-\mu-1}\left[\frac{(4)^{-(\nu-\alpha_2+j)}\gamma_{2j}(t')}{\nu-\alpha_2(t')+j}-\frac{(4)^{-(\nu-\alpha_2^*+j)}\gamma_{2j}^*(t')}{\nu-\alpha_2^*(t')+j}\right]. \quad (50)$$

The integers n and n' are determined by the inequalities

$$\tfrac{1}{2}>\sup_s(\mathrm{Re}\alpha_1(s)-n)>-\tfrac{1}{2},$$
$$\tfrac{1}{2}>\sup_t(\mathrm{Re}\alpha_2(t)-n')>-\tfrac{1}{2}.$$

The functions γ_{1j} and γ_{2j} are the new residues and are all proportional to β_1 and β_2. An expression for these functions in terms of β and α is given in the Appendix. We again assume that not only the β's but also the γ's vanish faster than $x^{-1/2}$ for large values of their arguments. This condition is necessary if one wants pure Regge asymptotic behavior and explicit crossing symmetry. We shall come back to a discussion of this point at the end of Sec. V. The function $c_0(\nu,\mu)$ is holomorphic in the region $\mathrm{Re}\nu>-\frac{1}{2}$, $\mathrm{Re}\mu>-\frac{1}{2}$. It is clear from (49) that $c(\nu,\mu;\alpha_1)$ is regular for $\mathrm{Re}\nu>-\frac{1}{2}$. However, in the μ plane it is regular everywhere except on the curves $\mu=\alpha_1(s)-j$, $j=0, 1, \cdots, n$, which are traced as s varies from 4 to infinity, and on the complex conjugate curves $\mu=\alpha_1^*(s)-j$. Similarly, $c(\nu,\mu;\alpha_2)$ is regular in μ for $\mathrm{Re}\mu>-\frac{1}{2}$, but in the ν plane we have to exclude the curves, $\nu=\alpha_2(t)-j$, and their complex conjugates. Although the properties of $c(\nu,\mu;\alpha_1)$ and $c(\nu,\mu;\alpha_2)$ seem rather complicated we shall see in a moment that when one substitutes (49) and (50) in (38) and carries out the summations one obtains just the contributions of the poles in a form identical to (24a,b).

It is not difficult to verify that, as $\mathrm{Re}\nu\to\infty$, $c_0(\nu,\mu)\sim(4)^{-\nu}$ for all μ with $\mathrm{Re}\mu>-\frac{1}{2}$; and that as $\mathrm{Re}\mu\to\infty$, $c_0(\nu,\mu)\sim(4)^{-\mu}$ for all ν with $\mathrm{Re}\nu>-\frac{1}{2}$. Similarly $c_0(\nu,\mu)$ vanishes when either $\mathrm{Im}\nu\to\pm\infty$ or $\mathrm{Im}\mu\to\pm\infty$ or both. The same type of asymptotic behavior holds for $c(\nu,\mu;\alpha_1)$ and $c(\nu,\mu;\alpha_2)$.

We thus have now all the conditions necessary to apply the Watson-Sommerfeld transformation to the double power series (38). For $|s|$, $|t|<4$ we have

$$f(s,t)=\sum_{\nu,\mu}c_0(\nu,\mu)s^\nu t^\mu+\sum_{\nu,\mu}c(\nu,\mu;\alpha_1)s^\nu t^\mu+\sum_{\nu,\mu}c(\nu,\mu;\alpha_2)s^\nu t^\mu. \quad (51)$$

[11] The upper limits of integration in (49) and (50) are at first s_1 and t_1, respectively. However, using our assumption about the asymptotic behavior of the γ's, we can make these limits ∞ by adding and subtracting terms regular in the ν and μ half-planes and absorbing the extra terms into $c_0(\nu,\mu)$.

For the first term one can easily apply the W-S formula twice and obtain

$$\sum_{\nu,\mu}c_0(\nu,\mu)s^\nu t^\mu=\frac{-1}{4}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\nu\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\mu\times\frac{c_0(\nu,\mu)}{\sin\pi\nu\,\sin\pi\mu}(-s)^\nu(-t)^\mu. \quad (52)$$

To sum the other two series in (51) we have to use (49) and (50). A typical term would be

$$R(t;\alpha_1(s)-j)=\sum_{\nu,\mu}s^\nu t^\mu\left[\frac{1}{2\pi i}\int_4^\infty ds'\,s'^{-\nu-1}\times\left(\frac{(4)^{-(\mu-\alpha_1+j)}\gamma_{1j}(s')}{\mu-\alpha_1(s')+j}-\frac{(4)^{-(\mu-\alpha_1^*+j)}\gamma_{1j}^*(s')}{\mu-\alpha_1^*(s')+j}\right)\right]. \quad (53)$$

The reason for using the same symbol for (53) as was used in (24a,b) will become apparent in the next few steps. Since $|s|<4$ we can easily perform the summation over ν and obtain

$$R(t;\alpha_1(s)-j)=\sum_{\mu=0}^{\infty}\frac{t^\mu}{2\pi i}\int_4^\infty\frac{ds'}{s'-s}\times\left\{\frac{(4)^{-(\mu-\alpha_1+j)}\gamma_{1j}(s')}{\mu-\alpha_1(s')+j}-\frac{(4)^{-(\mu-\alpha_1^*+j)}\gamma_{1j}^*(s')}{\mu-\alpha_1^*(s')+j}\right\}. \quad (54)$$

As we have already assumed that $\gamma_{1j}(s)$ vanish as $s\to\infty$, and using the fact that $\alpha_1(s)$ and $\gamma_{1j}(s)$ are analytic in the cut s plane, we can perform the integration over s' by closing the contour with a large circle in the s plane and get

$$R(t;\alpha_1(s)-j)=\sum_\mu t^\mu\left[\frac{(4)^{-(\mu-\alpha_1(s)+j)}\gamma_{1j}(s)}{\mu-\alpha_1(s)+j}\right];\quad \mu=0, 1, 2, \cdots. \quad (55)$$

Here we have made use of the fact that there are no ghosts and no bound states. Otherwise, we would have extra terms in (55) coming from those values of s on the physical sheet for which $\mu-\alpha_1(s)+j$ vanishes (μ and j are integers). The function $\alpha_1(s)$ will have a nonvanishing imaginary part for all values of s on the physical sheet except if s is real and lies in the interval $-\infty<s<4$. If for any $s=s_g$, $-\infty<s_g<0$, $\alpha_1(s_g)$ is a positive integer or zero, then that will correspond to a ghost pole. Following Gell-Mann[12] we assume that in that case $\beta(s_g)$ is zero as he demonstrated for the Pomeranchuk trajectory. If for $s=s_b$, $0<s_b<4$, $\alpha_1(s_b)$ is an integer or zero, then that would correspond to a bound state at $s=s_b$ and a pole in $f(s,t)$ at that point.

[12] M. Gell-Mann, in *Proceedings of the 1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), pp. 533–542.

We are excluding such poles for the present since in (31) we clearly started with an $f(s,t)$ with only branch cuts. At the end of this section we shall come back to the case with bound states.

Finally, we must stress that strictly speaking in all this section we should have used $\alpha_1(s-i\epsilon)$ instead of $\alpha_1^*(s)$ and $\gamma_{1j}(s-i\epsilon)$ instead of $\gamma_{1j}^*(s)$. At least the integral (54) over s' should be understood in that sense.

The series (55) is identical with (25) and the notation is justified. In fact one can now easily sum (55) by using the W-S transformation and the representation (19). One obtains

$$R(t;\alpha_1(s)-j)=\gamma_{1j}(s)\int_4^\infty \frac{x^{(\alpha_1-j)}}{x-t-i\epsilon}dx;$$

$$\mathrm{Re}(\alpha_1(s)-j)<-\tfrac{1}{2};\quad (56a)$$

or

$$R(t;\alpha_1(s)-j)=-\gamma_{1j}(s)\left[\int_0^4 \frac{x^{(\alpha_1-j)}}{x-t-i\epsilon}dx+\frac{\pi(-t)^{\alpha_1-j}}{\sin\pi(\alpha_1-j)}\right];$$

$$\mathrm{Re}(\alpha_1(s)-j)>-\tfrac{1}{2}.\quad (56b)$$

The results are identical with (24a,b) for the contribution of a pole at $\nu=\alpha_1-j$ in the case of a single-power series expansion. The expressions for the new residues γ_{1j} in terms of β_1 and α_1 are also given in the Appendix. The function R is easily recognized as an incomplete beta function times $\gamma_{1j}(s)$.

We can now write for $f(s,t)$

$$f(s,t)=-\frac{1}{4}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\nu \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\mu \frac{c_0(\nu,\mu)}{\sin\pi\nu\,\sin\pi\mu}(-s)^\nu(-t)^\mu$$

$$+\sum_{j=0}^{n} R(t;\alpha_1(s)-j)+\sum_{j=0}^{n'} R(s;\alpha_2(t)-j).\quad (57)$$

The integers n and n' are determined by the inequalities given below (49) and (50). The functions R take the form (56a) if $\mathrm{Re}(\alpha_{1,2}-j)<-\frac{1}{2}$ and (56b) if $\mathrm{Re}(\alpha_{1,2}-j)>-\frac{1}{2}$.

So far we have been holding s and t in the region $|s|, |t|<4$. We can now continue the right-hand side of (57) into the cut s and t planes. The first term seems to have cuts starting at $s=0$ and $t=0$; however this is not the case as we shall see below. The functions R satisfy a Mandelstam-type representation similar to (31) and if in varying s or t, $(\alpha_{1,2}-j)$ moves from the right-half to the left-half plane R goes in an analytic way from the form (56b) to (56a) or vice versa. Note that (56a) and (56b) are identical in the strip $-1<\mathrm{Re}(\alpha_{1,2}-j)<0$.

To show that the first term in (57) has the right cuts we use (19) twice and obtain

$$f(s,t)=\int_0^\infty dx\int_0^\infty dy\frac{b(x,y)}{(x-s)(y-t)}+\sum_{j=0}^{n} R(t;\alpha_1(s)-j)$$

$$+\sum_{j=0}^{n'} R(s;\alpha_2(t)-j),\quad (58)$$

where

$$b(x,y)=\frac{1}{(2\pi i)^2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\nu\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\mu\, x^\nu y^\mu c_0(\nu,\mu).\quad (59)$$

Now using the fact that $c_0(\nu,\mu)\sim(4)^{-\nu}$ for large $\mathrm{Re}\nu$ and that it vanishes for large $\mathrm{Im}\nu$ we can for $x<4$ close the contour of the ν integration to the right and obtain

$$b(x,y)=0,\quad x<4,\quad y>0;\quad (60)$$

and in a similar manner

$$b(x,y)=0,\quad y<4,\quad x>0.\quad (60')$$

It is also clear from (59) and the fact that $c_0(\nu,\mu)$ vanishes when either $\mathrm{Im}\nu$ or $\mathrm{Im}\mu$ become infinite, that that $b(x,y)$ vanishes faster than $x^{-1/2}$ (or $y^{-1/2}$) as either argument becomes infinite. The integrals in (59) are defined as contour integrals.

We thus finally get,

$$f(s,t)=\int_4^\infty dx\int_4^\infty dy\frac{b(x,y)}{(x-s-i\epsilon)(y-t-i\epsilon)}$$

$$+\sum_{j=0}^{n} R(t;\alpha_1(s)-j)+\sum_{j=0}^{n'} R(s;\alpha_2(t)-j).\quad (61)$$

The first term above never needs any subtractions. One could write Mandelstam-type double-dispersion integrals for the R functions but these however will need subtractions. We have thus explicitly factored out the subtractions in terms of Regge pole contributions.

In the process of deriving (61) we have made several assumptions about $f(s,t)$ and about the β's and α's. Most of these assumptions are natural in the sense that they have either been proved for the relativistic scattering problem or at least conjectured on the basis of analogy to potential scattering. The main new assumption in the present discussion is the limitation that the residues $\gamma(x)$ vanish at least as fast as $x^{-1/2}$ for large x. This assumption is necessary if Regge behavior is to be made consistent with explicit crossing symmetry. We shall come back to a more detailed discussion of this point at the end of the next section.

One condition which we have imposed on $f(s,t)$ which we would like now to relax is the absence of bound-state type poles. Suppose now that f has a bound-state pole at $s=s_b$ and $0<s_b<4$. We shall sketch below how the preceding discussion has to be modified to obtain the representation (61).[13]

In the first place if we have such a pole the representation (31) will have to be written as

$$f(s,t)=\Gamma/(s-s_b)+L(s,t),$$

with (31′)

$$L(s,t)=\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{\rho(s',t')}{(s'-s)(t'-t)}. \quad (31')$$

We have taken here an S-wave pole for simplicity. It is clear now that in this case $\alpha_1(s_b)=0$, and the residue Γ is related to β by $\Gamma=-\beta(s_b)/\alpha'(s_b)$.

We now expand $L(s,t)$ instead of $f(s,t)$ in double-power series as in (38),

$$L(s,t)=\sum_{\nu,\mu} c(\nu,\mu)s^\nu t^\mu. \quad (38')$$

The expression for $c(\nu,\mu)$ will be unchanged from the previous case. The derivation follows identical steps until we reach the point where we are summing the series $\sum c(\nu,\mu;\alpha_1)s^\nu t^\mu$, as in Eqs. (53) to (55). Here we get an extra term in (55) which comes from contribution of the pole in the s plane resulting from the fact that $[\mu-\alpha_1(s)+j]$ for $\mu=j=0$ vanishes when $s=s_b$. This extra term will be given by $(\gamma_{10}(s_b)/\alpha'(s_b))(s-s_b)^{-1}$. From the Appendix we find that $\gamma_{10}(s_b)=\beta(s_b)$, since $\alpha(s_b)=0$. Thus, when we substitute back into (31′) this extra term will exactly cancel the first term of (31′) and we end up with an expression for $f(s,t)$ identical to (61). The bound-state pole at $s=s_b$ now shows up in one of the R functions.

A similar cancellation holds for higher l bound states or for bound state poles in the t channel.

V. CROSSING-SYMMETRIC WATSON-SOMMERFELD TRANSFORMATION

In this section we shall use the results of the previous section to obtain a crossing-symmetric Regge representation for the invariant scattering amplitude in a full-relativistic problem. We consider the case of equal-mass, neutral, spinless particles. We start by writing down the Mandelstam representation for the amplitude $A(s,t,u)$,

$$A(s,t,u)=L_{12}(s,t)+L_{23}(t,u)+L_{13}(s,u), \quad (62)$$

where

$$L_{12}(s,t)=\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{\rho_{12}(s',t')}{(s'-s)(t'-t)}; \quad (63)$$

similar expressions hold for L_{23} and L_{13}. In (63) we have not written any subtraction terms. This as we shall see below is not necessary. Martin[14] has recently proved, under assumptions that are weaker than the ones we are going to assume, that the weight functions ρ_{ij} uniquely determine $A(s,t,u)$. In fact, our discussion will give another demonstration of this fact. Again to simplify the algebra we consider the case where there are no single-particle or bound-state poles in the s, t, or u planes. As in the discussion at the end of the previous section, such poles can be included by a small modification of the analysis below and the result will not change.

In the s channel we have the following relations that define the cosine of the c.m. scattering angle, z_1, and the c.m. momentum q_1,

$$\begin{aligned} 4q_1^2&=s-4\\ z_1&=1+2t/(s-4)\\ -z_1&=1+2u/(s-4). \end{aligned} \quad (64)$$

We shall consistently use the indices 1, 2, 3 to denote the s, t, u channels, respectively. For each of the other two channels there are relations like (64) which can be obtained from it by permutting the variables s, t, u.

As usual one separates $A(s,z_1)$ into even and odd parts in z_1 and write the partial-wave expansion for the s channel as,

$$A^{(\pm)}(s,z_1)=\sum_l (2l+1)\frac{a_1^{(\pm)}(l,s)}{2}\times\left[P_l\left(1+\frac{2t}{s-4}\right)\pm P_l\left(1+\frac{2u}{s-4}\right)\right]. \quad (65)$$

Two other expansions can be written for the t channel and the u channel, respectively.

Proceeding as in the previous section we now assume that the $a_j^{(\pm)}$; $j=1, 2, 3$; can each be analytically continued into the right-half l plane, $\mathrm{Re}l>-\frac{1}{2}$, except for poles which might show up in that region. Following Oehme,[15] we consider only moving poles. For simplicity we again take only one pole in each channel and choose that pole to be of positive signature, i.e., a pole in $a_j^{(+)}$. For example, in the s channel we assume that $a_1^{(+)}(l,s)$ has a pole at $l=\alpha_1(s)$ such that

$$\mathrm{Re}\alpha_1(s)>-\tfrac{1}{2},\quad s_0<s<s_1;\ s_0<4;\ s_1>4; \quad (66)$$

and $\mathrm{Re}\alpha_1(s)<-\frac{1}{2}$ for any real s outside the above interval. We assume that as s varies from $-\infty$ to $+\infty$ $a_1^{(-)}(l,s)$ has no poles that show up in the region $\mathrm{Re}l>-\frac{1}{2}$. This last assumption is made just to simplify the algebra and is not necessary. We make similar assumptions about $a_2^{(\pm)}(l,t)$ and $a_3^{(\pm)}(l,u)$, introducing two more poles $\alpha_2(t)$ and $\alpha_3(u)$. For each of these poles we have inequalities like (66) holding. Finally, we assume that each of the amplitudes $a_j^{(\pm)}$ for large $|l|$ behaves as in (37).

[13] The author is indebted to S. B. Treiman and R. Blankenbecler for helpful remarks on this point.

[14] A. Martin, Phys. Rev. Letters 9, 410 (1962).

[15] R. Oehme, Phys. Rev. Letters 9, 358 (1962).

Instead of expanding the whole amplitude $A(s,t)$ in a double series in s and t, we write A as the sum of three double series one corresponding to each of the terms in (62). We write

$$A(s,t,u)=\sum_{\nu,\mu} c_{12}(\nu,\mu)s^\nu t^\mu+\sum_{\nu,\mu} c_{23}(\nu,\mu)t^\nu u^\mu +\sum_{\nu,\mu} c_{13}(\nu,\mu)s^\nu u^\mu. \quad (67)$$

The series given here will converge absolutely for s, t, u inside the Euclidean region, $0<s, t, u<4$ and $s+t+u=4$.

Following the previous section we can now write the c_{ij} as double Mellin transforms of the corresponding ρ_{ij},

$$c_{ij}(\nu,\mu)=\frac{1}{\pi^2}\int_4^\infty dx\int_4^\infty dy\rho_{ij}(x,y)x^{-\nu-1}y^{-\mu-1}, \quad i, j=1, 2, 3. \quad (68)$$

From here on the argument is identical with that of Sec. IV. We shall sketch it briefly, stressing the differences.

It follows from (68) that $c_{ij}(\nu,\mu)$ is holomorphic in the region

$$\mathrm{Re}\nu>\sup_x[\mathrm{Re}\alpha_j(x)],$$
$$\mathrm{Re}\mu>\sup_x[\mathrm{Re}\alpha_i(x)].$$

To extend the region of analyticity further to the left we have to use representations for ρ_{ij} analogous to (41) and (42). Applying the Watson-Sommerfeld transformation to the s-channel partial-wave expansion (65), and taking the double discontinuity in s and t, we get

$$\rho_{12}(s,t)=\sigma_{12}^{(1)}(s,t)+\frac{\pi}{4i}\left[\beta_1(s)P_{\alpha_1}\left(1+\frac{2t}{s-4}\right) -\beta_1^*(s)P_{\alpha_1^*}\left(1+\frac{2t}{s-4}\right)\right], \quad 4<s<s_1; t>4. \quad (69)$$

Here again we have

$$\sigma_{12}^{(1)}(s,t)\sim t^{-1/2}; \quad t\to\infty; \quad s>4. \quad (70)$$

Similarly by taking the double discontinuity in s and u we obtain

$$\rho_{13}(s,u)=\sigma_{13}^{(1)}(s,u)+\frac{\pi}{4i}\left[\beta_1(s)P_{\alpha_1}\left(1+\frac{2u}{s-4}\right) -\beta_1^*P_{\alpha_1^*}\left(1+\frac{2u}{s-4}\right)\right]; \quad 4<s<s_1; u>4. \quad (71)$$

The first term satisfies

$$\sigma_{13}^{(1)}(s,u)\sim u^{-1/2}; u\to\infty; \quad s>4. \quad (72)$$

From the partial-wave expansions in the t and u channels we get four more relations analogous to (69) and (71), two from each channel. Thus, for each ρ_{ij} one obtains two relations similar to (41) and (42). One of these relations exhibits the poles of the i channel and the other the poles of the j channel. Again, results similar to (44) are valid for each ρ_{ij}.

It becomes evident at this stage that the situation for each $c_{ij}(\nu,\mu)$ is completely analogous to that of the previous section and almost identical results follow.

We first obtain

$$c_{ij}(\nu,\mu)=c_{ij}^{(0)}(\nu,\mu)+\tfrac{1}{2}c_{ij}(\nu,\mu;\alpha_i)+\tfrac{1}{2}c_{ij}(\nu,\mu;\alpha_j), \quad i, j=1, 2, 3. \quad (73)$$

Here $c_{ij}^{(0)}(\nu,\mu)$ is holomorphic in the domain $\mathrm{Re}\nu>-\frac{1}{2}$, $\mathrm{Re}\mu>-\frac{1}{2}$. The other two functions, $c_{ij}(\nu,\mu;\alpha_i)$ and $c_{ij}(\nu,\mu;\alpha_j)$ are defined by (49) and (50), respectively, and have corresponding analytic properties.

We can now apply the Watson-Sommerfeld transformation to each of the three series in (67), keeping s, t, u in the Euclidean region, and we obtain

$$A(s,t,u)=-\frac{1}{4}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\nu\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\mu\frac{1}{\sin\pi\nu\,\sin\pi\mu}[c_{12}^{(0)}(\nu,\mu)(-s)^\nu(-t)^\mu+c_{23}^{(0)}(\nu,\mu)(-t)^\nu(-u)^\mu+c_{13}^{(0)}(\nu,\mu)(-s)^\nu(-u)^\mu]$$
$$+\sum_{r=0}^{n_1}(\tfrac{1}{2})[R(t;\alpha_1(s)-r)+R(u;\alpha_1(s)-r)]+\sum_{r=0}^{n_2}(\tfrac{1}{2})[R(u;\alpha_2(t)-r)+R(s;\alpha_2(t)-r)]$$
$$+\sum_{r=0}^{n_3}(\tfrac{1}{2})[R(s;\alpha_3(u)-r)+R(t;\alpha_3(u)-r)]. \quad (74)$$

The functions R are defined in (56a,b). The integers n_i, $i=1, 2, 3$, are determined by the inequalities

$$\tfrac{1}{2}>\sup_x(\mathrm{Re}\alpha_i(x)-n_i)>-\tfrac{1}{2}. \quad (75)$$

We can now continue the right-hand side of (74) for values of s, t, u outside the Euclidean region, always keeping $s+t+u=4$. As in the previous case the first term can be written as the sum of three Mandelstam-type integrals with the correct cuts and no subtractions. The argument is identical to that of (57)–(61), and we get

$$A(s,t,u)=\int_4^\infty dx\int_4^\infty dy\frac{b_{12}(x,y)}{(x-s)(y-t)}+\int_4^\infty dx\int_4^\infty dy\frac{b_{23}(x,y)}{(x-t)(y-u)}+\int_4^\infty dx\int_4^\infty dy\frac{b_{13}(x,y)}{(x-s)(y-u)}$$

$$+\sum_{r=0}^{n_1}[R(t;\alpha_1(s)-r)+R(u;\alpha_1(s)-r)](\tfrac{1}{2})+\sum_{r=0}^{n_2}[R(u;\alpha_2(t)-r)+R(s;\alpha_2(t)-r)](\tfrac{1}{2})$$

$$+\sum_{r=0}^{n_3}[R(s;\alpha_3(u)-r)+R(t;\alpha_3(u)-r)](\tfrac{1}{2}).\quad(76)$$

The functions $b_{ij}(x,y)$ are defined by

$$b_{ij}(x,y)=\left(\frac{1}{2\pi i}\right)^2\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\nu\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}d\mu x^\nu y^\mu c_{ij}^{(0)}(\nu,\mu).\quad(77)$$

We also have the result that $b_{ij}(x,y)=0$ when either $x<4$ or $y<4$. Furthermore, $b_{ij}(x,y)$ vanish when either argument becomes infinite.

The representation (76) has some interesting features other than its explicit crossing symmetry. We originally started in (62) with a Mandelstam representation which had subtraction terms even though we did not explicitly write them down. We ended up in (76) with an expression that has three Mandelstam-type terms that need no subtractions plus the full contributions of all the physical Regge poles in all three channels. By physical Regge poles we mean those poles which for some real interval of s, t, or u lie in the region $\mathrm{Re}l>-\frac{1}{2}$.[16]

Under the assumption we have made about the asymptotic behavior of the residues $\gamma_{ir}(x)$, it is easy to check that (76) will lead to the usual Regge-type asymptotic behavior for all channels. For example, for large positive s and t near zero (i.e., u large and negative), (76) will have a term proportional to $\gamma_{20}(t)\times(-s)^{\alpha_2(t)}$ which gives the usual Regge behavior. However, there will also be terms proportional to $\gamma_{1r}(s)$ and unless these residues vanish faster than $s^{-1/2}$ we would end up with an additional non-Regge-type asymptotic term coming from these residues. Of course, in our derivation we made use of the condition $\gamma_{ir}(x)\sim x^{-1/2}$ as $x\to\infty$. If, on the other hand, someone had obtained (76) via another procedure, it is clear that the crossing symmetric result in (76) will be in contradiction with pure Regge behavior if $\gamma_{ir}(x)\sim x^a$ as $x\to\infty$ and $a>-\frac{1}{2}$. Thus, our restriction on the γ's seems to be unavoidable. A similar statement could be made about the necessity of having the trajectories move back into the region $\mathrm{Re}l<-\frac{1}{2}$ for large positive or negative s, t, or u.

Near a resonance when $\mathrm{Re}\alpha_i\approx n$, n a positive integer or zero, and $\mathrm{Im}\alpha_i$ is small, the R functions in (76) just combine to give us the usual Regge term proportional to $\beta_i P_{\alpha_i}(z)/\sin\pi\alpha_i$.

In conclusion, we would like to stress that in this paper we have been mainly interested in exploring some of the consequences of the analytic properties of the partial-wave amplitude as a function of complex angular momentum l. Mainly we have shown the close relationship between these properties and the analytic properties of the coefficients of expansions other than the partial-wave expansion as functions of their respective indices. For simplicity of the discussion we have started by assuming the simplest type of singularities in the l plane, i.e., moving poles. However, results similar to, but more complicated than, those derived in this paper could in general be obtained if one starts with more involved singularities in the l planes (poles and cuts). Of course, again in that case one will have to impose certain conditions on the nature of these cuts.

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APPENDIX

In this short Appendix we shall give a few of the formulas which relate the new residues γ_{ir} to β_i and α_i. Here we note that α_i stands for the position of the pole in the l plane and β_i is the residue (times $[2\alpha_i+1]$) of the partial-wave amplitude at $l=\alpha_i$. The formulas are the same for both the relativistic and nonrelativistic case except for the fact that in the first case $q^2=\frac{1}{4}(s-4)$ and in the second $q^2=s$.

[16] We do not include here any threshold poles. Namely, those poles which for energies very near threshold lie arbitrarily close to the line $\mathrm{Re}l=-\frac{1}{2}$ and have $\mathrm{Re}l>-\frac{1}{2}$, but which move back quickly into $\mathrm{Re}l<-\frac{1}{2}$ as we increase the energy from threshold. The contributions of such poles could be easily absorbed into the background terms if instead of integrating along the lines $\mathrm{Re}\nu=-\frac{1}{2}$ we use the line $\mathrm{Re}\nu=-\frac{1}{2}+\epsilon$ with ϵ chosen such that all threshold poles lie to the left of $\mathrm{Re}l=-\frac{1}{2}+\epsilon$.

The results are

$$\gamma_{i0}=\frac{\beta_i}{(2q^2)^{\alpha_i}}\frac{2^{\alpha_i}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\alpha_i)}{\Gamma(1+\alpha_i)}. \tag{A1}$$

$$\gamma_{i1}=\frac{\beta_i}{(2q^2)^{\alpha_i-1}}\frac{2^{\alpha_i}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\alpha_i)}{\Gamma(\alpha_i)}. \tag{A2}$$

$$\gamma_{i2}=\frac{\beta_i}{(2q^2)^{\alpha_i-2}}\frac{2^{\alpha_i}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\alpha_i)}{2!\Gamma(\alpha_i-1)},\quad -\tfrac{1}{2}<\mathrm{Re}\alpha_i<\tfrac{3}{2};$$

$$=\frac{\beta_i}{(2q^2)^{\alpha_i-2}}\frac{2^{\alpha_i}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\alpha_i)}{2!\Gamma(\alpha_i-1)}+\frac{\beta_i}{(2q^2)^{\alpha_i-2}}\frac{2^{\alpha_i-2}}{\sqrt{\pi}}\times\frac{\alpha_i(\alpha_i-1)}{(\frac{1}{2}-\alpha_i)}\frac{\Gamma(\alpha_i+\frac{1}{2})}{\Gamma(\alpha_i+1)},\quad \mathrm{Re}\alpha_i>\tfrac{3}{2}. \tag{A3}$$

The other γ_{ir} for $r=3, 4, \cdots, n_i$ can be computed by finding the residue of the function $I(\alpha_i,\nu)$ at the pole $\nu=\alpha_i-r$. As in (15) the function $I(\alpha_i,\nu)$ is given by

$$I(\alpha_i,\nu)=\beta_i\int_{t_0}^{\infty}P_{\alpha_i}\left(1+\frac{t}{2q^2}\right)t^{-\nu-1}dt,\quad \mathrm{Re}\alpha_i>-\tfrac{1}{2}. \tag{A4}$$

In the discussion in Secs. III and IV we have assumed that $\gamma_{ir}(s)$ are regular in the cut s plane. However, if we continue (A1)–(A3) in s the gamma functions $\Gamma(\frac{1}{2}+\alpha_i(s))$ will have a pole whenever s is such that α_i is negative and half-odd integral. For the simplicity of the discussion we did assume that we have a ghost-killing mechanism similar to that of Gell-Mann mentioned below (55). We took $\beta_i(s)$ to be zero for those values of s for which $\alpha_i(s)$ is a negative half-odd integer. We recall that under the usual analytic properties of $\alpha_i(s)$ this function is real for s on the physical sheet only if s is real and $-\infty<s<4$.

The imposition of this condition on $\beta_i(s)$ is not really necessary. For the poles of $\gamma_{ir}(s)$ resulting from the Γ functions in (A1–3) do not lead to ghost poles of the amplitude in the s plane. To see this let $\alpha_1(s_1)=-(2n+1)/2$ where n is an integer. In that case $\gamma_{1r}(s)$ will have a pole at $s=s_1$. This will lead to an additional term in (55) and the contribution of the Regge pole will be now

$$R'=R(t;\alpha_1(s)-r)-\frac{f_{1r}}{s-s_1}\int_4^{\infty}\frac{x^{-(n+\frac{1}{2})-r}}{x-t}dx. \tag{A5}$$

Here f_{1r} is the residue of γ_{1r} at $s=s_1$. Both terms in (A5) have poles at $s=s_1$. However, it is clear from (56b) that these poles exactly cancel each other and the amplitude will have no ghost at $s=s_1$. We can use R' instead of R in (61) and (76).

Finally, we mention that the factors $\Gamma(\frac{1}{2}+\alpha)$ also appear in Gell-Mann's expressions for the residue quantities η in Ref. 12, pp. 537 and 538. There again one presumably assumes that the pole in Γ as $\alpha\to-\frac{1}{2}(2n+1)$ is destroyed by a zero in the other factors. Both in the present case and in that of reference 11 the pole in Γ for $\alpha=-\frac{1}{2}$ is destroyed by a $(2\alpha+1)$ factor which is included in β. The problem discussed in the last three paragraphs here will thus not arise in the case of trajectories for which $\alpha(s)>-\frac{3}{2}$ for all s in the interval $-\infty<s<4$.